A new class of integrable defects

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Abstract
An alternative Lagrangian definition of an integrable defect is provided and
analysed. The new approach is sufficiently broad to allow a description
of defects within the Tzitzéica model, which was not possible in previous
approaches, and may be generalizable. New, two-parameter, sine-Gordon
defects are also described, which have characteristics resembling a pair of
'fused' defects of a previously considered type. The relationship between
these defects and Bäcklund transformations is described and a Hamiltonian
description of integrable defects is proposed.

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1. Introduction

It was noted some years ago [1, 2] that an integrable field theory in two-dimensional spacetime
can accommodate discontinuities yet remain integrable. The fields on either side of a
discontinuity are related to each other by a set of 'defect' conditions, including the influence of
a 'defect' potential whose form is required by integrability. The defect conditions themselves
are interesting since they are related, at least in the examples investigated so far, to Bäcklund
transformations frozen at the location of the defect. It has been found, possibly owing
ultimately to the latter observation, that defects can be supported within the $a_n^{(1)}$ series of
affine Toda models [3, 4], of which the sine-Gordon model is the first member. Intriguingly,
and despite translation invariance being explicitly broken by the prescribed location, defect
conditions compatible with integrability are determined simply by demanding that the defect
itself be able to contribute consistently to ensure the whole system supports a conserved
energy and momentum. The defect may be located anywhere (or even move at a constant
speed [5]), but the defect conditions apparently compensate for the evident lack of translation
invariance. One might regard the defect as a state within the model whose presence is indicated
by a set of defect conditions described by an additional term in the Lagrangian description
rather than being a field excitation or smooth field configuration. Typically, an integrable
defect will be purely transmitting and its effect does not depend upon its location, meaning

it is essentially 'topological'. At a classical level this is exemplified by the passage of a sine-Gordon soliton through a defect where the soliton will be delayed (or advanced), but might alternatively, according to circumstances, be absorbed by the defect or flipped to an anti-soliton [5]. Similar types of behaviour are observed for the complex solitons of the $a_n^{(1)}$ models [6]. At a quantum level, defects also appear to play a role though again they are purely transmitting and described by a transmission matrix that is compatible with the bulk scattering matrix. The purely transmitting aspect of the setup was to be expected from observations by Delfino *et al* [7, 8], but it is still of interest to see exactly how this transpires in detail. In the sine-Gordon case, the transmission matrix was anticipated by Konik and LeClair [9] but rederived and its properties explored in detail in [5]; for other members of the $a_n^{(1)}$ series, the transmission matrices have been provided more recently [10]. There are a number of related ideas and calculations, including perturbative checks of transmission factors for breathers, and an analysis of the interesting relationship between integrable boundary conditions and defects; some of these are explored in the article by Bajnok and Simon [11].

The sine-Gordon Bäcklund transformation was generalized to $a_n^{(1)}$ affine Toda models by Fordy and Gibbons [12] and it seems surprising there appear to be no similarly explicit Bäcklund transformations for the other series of Toda models. However, that fact is at least consistent with the apparent absence of defects in most of these models, at least of the kind previously considered [10]. On the other hand, there are several types of Bäcklund transformation available in the literature for the Tzitzéica model [13–16][1] and, therefore, one might suppose that there should be a generalization of the defect, at least for this model, and possibly for others. The purpose of this paper is to propose a generalization by allowing a defect to have its own degree of freedom in a certain well-defined manner, which is just general enough to encompass the Tzitzéica model. This is reminiscent of an idea of Baseilhac and Delius concerning dynamical boundaries [17, 18] though it turns out to be rather different in practice. Applying the same idea to massive free fields and to the sine-Gordon model leads to new types of defect even there, encouraging the possibility of finding a more general framework that might be able to accommodate defects in all Toda models. It is interesting also to note that in the sine-Gordon model the new defects belong to a two-parameter family, which in a certain sense might be regarded as 'bound states' of the defects introduced in [1].

As mentioned above, the requirement of overall energy–momentum conservation is surprisingly powerful and will be the main technique employed, although, clearly, further checks are needed to verify integrability. On the other hand, previous experience strongly suggests that the conditions following from momentum conservation in the presence of a defect are more or less equivalent to the restrictions imposed by integrability: for example, even if the bulk models on either side of the defect are not specified in advance, they will be severely restricted by insisting on momentum conservation once the defect is taken into account. So far, unlike the cases within the older framework, where the integrability is underpinned by a generalized Lax pair [1, 2], no suitable Lax pair description describing the new framework exists yet, and it is necessary to provide alternative arguments. A small step in this direction is provided in appendix A where it is demonstrated that the new defect conditions for the sinh/sine-Gordon equation are enough to ensure the existence of a conserved spin three charge. Other, indirect, evidence is provided in section 5 where the relationships between defects of different types and Bäcklund transformations are elaborated. Finally, a sketch of a Hamiltonian approach is given in

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[1] *Note*: the model introduced by Tzitzéica is the $a_2^{(2)}$ member of the affine Toda collection of field theories and is also known as the Bullough–Dodd or Zhiber–Mikhailov–Shabat equation.

section 6 within which defect conditions are regarded as constraints imposed at the location of the defect on the fields to either side of it.

2. Generalizing the framework

Consider a defect located at the origin $x = 0$ and let u and v be the fields on either side of it in the regions $x < 0$ and $x > 0$, respectively. Typically, a defect defined by Bäcklund conditions will have a discontinuity, in the sense that while the conditions sewing the two fields at the origin constrain their derivatives the fields themselves are not prescribed. In other words, it is expected that the values of the fields approaching $x = 0$ from their respective domains need not match and it should be expected that $u(0, t) - v(0, t) \neq 0$. The basic idea to be explored here introduces a new variable $\lambda(t)$ associated with the defect itself. The simplest setup one might envisage does not directly associate dynamics with λ but is linear in λ_t having a Lagrangian description of the form:

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(\frac{uv_t - vu_t}{2} + \lambda(u - v)_t - \lambda_t(u - v) - \mathcal{D}(u, v, \lambda)\right), \quad (2.1)$$

where the Heaviside step function $\theta(x)$ and the Dirac delta function have been inserted to ensure the fields u, v are restricted to their respected domains with the defect located at $x = 0$. In a sense, $\lambda(t)$ plays the role of a Lagrange multiplier: if the potential were absent, integrating over λ would require the discontinuity to be time independent. However, because the potential also depends on λ it has a more interesting effect. As we shall see, this is the case even if the potential is quadratic and the defect links two free massive fields. For the purposes of distinguishing the cases with and without the extra degree of freedom, defects of the original type ($\lambda \equiv 0$) will be called type I and those where λ plays a role will be called type II.

The defect conditions at $x = 0$ implied by (2.1) are

$$u_x = v_t - 2\lambda_t - \frac{\partial \mathcal{D}}{\partial u}, \quad (2.2)$$

$$v_x = u_t - 2\lambda_t + \frac{\partial \mathcal{D}}{\partial v}, \quad (2.3)$$

$$u_t = v_t + \frac{1}{2}\frac{\partial \mathcal{D}}{\partial \lambda}. \quad (2.4)$$

Then, it is not difficult to show directly that $\mathcal{E} + \mathcal{D}$ is conserved, where $\mathcal{E}$ is the combined bulk contributions to the total energy from the fields u and v. This was to be expected since time translation invariance has not been violated.

On the other hand, as usual the contribution from the fields u and v to the total momentum is not conserved and the requirement of being able to construct a compensating contribution from the defect is highly constraining. Defining

$$\mathcal{P} = \int_{-\infty}^{0} dx \, u_x u_t + \int_{0}^{\infty} dx \, v_x v_t, \quad (2.5)$$

differentiating with respect to time, and using the bulk equations of motion, gives

$$\dot{\mathcal{P}} = \tfrac{1}{2}\left(u_t^2 + u_x^2 - 2U(u)\right)_{x=0} - \tfrac{1}{2}\left(v_t^2 + v_x^2 - 2V(v)\right)_{x=0}. \quad (2.6)$$

Using the defect conditions (and simplifying the notation on the understanding all field quantities are evaluated at $x = 0$), the latter can be rewritten as

$$-v_t \frac{\partial \mathcal{D}}{\partial u} - u_t \frac{\partial \mathcal{D}}{\partial v} + 2\lambda_t \left(\frac{\partial \mathcal{D}}{\partial u} + \frac{\partial \mathcal{D}}{\partial v} + \frac{1}{2}\frac{\partial \mathcal{D}}{\partial \lambda}\right) + \frac{1}{2}\left(\left(\frac{\partial \mathcal{D}}{\partial u}\right)^2 - \left(\frac{\partial \mathcal{D}}{\partial v}\right)^2\right) - U + V. \quad (2.7)$$

For type I defects it would be natural to require the last piece (without any time derivatives) to vanish and the first two pieces should be a total time derivative leading to equations for the potential $\mathcal{D}$:

$$\frac{\partial^2 \mathcal{D}}{\partial u^2} = \frac{\partial^2 \mathcal{D}}{\partial v^2}, \qquad \frac{1}{2}\left(\left(\frac{\partial \mathcal{D}}{\partial u}\right)^2 - \left(\frac{\partial \mathcal{D}}{\partial v}\right)^2\right) = U - V. \tag{2.8}$$

This was the setup originally considered in [1]. In fact, as was recalled in the introduction, conditions (2.8) are highly constraining, effectively limiting U, V (and $\mathcal{D}$) to the set of sine/sinh-Gordon, Liouville, massive or massless, free fields. In particular, the Tzitzéica equation is explicitly excluded. It is also worth recalling the well-known fact that the same selection of fields follows from insisting on the conservation of a spin-3 charge in the bulk (and that a careful analysis of the energy-like spin three charge is enough to provide the full set of integrable boundary conditions for the sine/sinh-Gordon model [19]). The Tzitzéica equation does not allow the conservation of a spin-3 charge but is the one additional possibility that arises if one instead examines a bulk conserved charge of spin five.

However, for type II defects, where $\lambda \neq 0$, the condition on the part of (2.7) containing no explicit derivatives is weaker because it need not be zero as was assumed in (2.8). Rather, it should be equated with

$$\frac{1}{2} F(u, v, \lambda) \frac{\partial \mathcal{D}}{\partial \lambda} \equiv (u - v)_t F(u, v, \lambda),$$

for some function F depending on u, v and λ, but not their derivatives. In turn, this observation modifies the impact of the other terms. Taking it into account and assuming the result is a total time derivative of $-\Omega$, designed to be a functional of $u(0, t)$, $v(0, t)$ and $\lambda(t)$, requires

$$\frac{\partial \Omega}{\partial u} = \frac{\partial \mathcal{D}}{\partial v} - F$$
$$\frac{\partial \Omega}{\partial v} = \frac{\partial \mathcal{D}}{\partial u} + F$$
$$\frac{\partial \Omega}{\partial \lambda} = -2\left(\frac{\partial \mathcal{D}}{\partial u} + \frac{\partial \mathcal{D}}{\partial v} + \frac{1}{2}\frac{\partial \mathcal{D}}{\partial \lambda}\right), \tag{2.9}$$
$$\left(\frac{\partial \mathcal{D}}{\partial u}\right)^2 - \left(\frac{\partial \mathcal{D}}{\partial v}\right)^2 = 2(U - V) + F\,\mathcal{D}_\lambda.$$

This set of equations entails a number of compatibility relations and to examine these it is convenient to use new field coordinates defined at the defect location:

$$p = \frac{u(0, t) + v(0, t)}{2}, \qquad q = \frac{u(0, t) - v(0, t)}{2}.$$

Then, after a few manipulations the conditions become (and hereafter subscripts will be used to denote partial derivatives):

$$\Omega_p = \mathcal{D}_p,$$
$$\Omega_q = -\mathcal{D}_q - 2F, \tag{2.10}$$
$$\Omega_\lambda = -\mathcal{D}_\lambda - 2\mathcal{D}_p.$$

Eliminating Ω leads to

$$\mathcal{D}_{pq} = -F_p,$$
$$\mathcal{D}_{\lambda p} = -\mathcal{D}_{pp}, \tag{2.11}$$
$$F_\lambda = -F_p,$$

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and, from these it follows that

$$\mathcal{D} = f + g, \qquad F = -f_q, \qquad \Omega = f - g,$$

where g depends only on λ and q, and f depends on q and $p - \lambda$. Under these circumstances, the last, nonlinear, relation becomes

$$\mathcal{D}_p \mathcal{D}_q = 2(U - V) + (f_\lambda + g_\lambda)F,$$

and this may also be rearranged and rewritten in terms of derivatives of f and g:

$$\tfrac{1}{2}(f_q g_\lambda - f_\lambda g_q) = U - V. \tag{2.12}$$

Interestingly, the left-hand side of (2.12) is equal to the Poisson bracket of f and g regarded as functions of λ and its conjugate momentum $\pi_\lambda = -(u - v) = -2q$. In terms of the defect energy and momentum, $\mathcal{D}$ and Ω, the relationship (2.12) is

$$\{\mathcal{D}, \Omega\} = -2(U - V), \tag{2.13}$$

an intriguing equation that relates the Poisson bracket of the energy and momentum contributed by the defect, which is non-zero because of the lack of translation invariance, to the potential difference across the defect.

Finally, it is worth noting that equation (2.12) is powerful because all the dependence on λ contained on the left-hand side of the equation must cancel out; this significantly constrains not only f and g but also the potentials $U(u)$ and $V(v)$. As will be seen below the list of possibilities will now include the Tzitzéica model that had been excluded previously.

3. Examples

In this section, using natural ansätze, a number of possible solutions to (2.12) are given. Besides the Tzitzéica equation these solutions provide generalizations of already known integrable defects. However, it is not clear that the examples given exhaust all possible solutions to (2.12).

3.1. The sinh/sine-Gordon model

For the sine-Gordon model, given the form of the potentials

$$U(u) = e^{p+q} + e^{-p-q} \equiv e^u + e^{-u}, \qquad V(v) = e^{p-q} + e^{-p+q} \equiv e^v + e^{-v},$$

and bearing in mind the form of the constraint (2.12), the most general ansatz for f and g is

$$f = A\,e^{p-\lambda} + B\,e^{-p+\lambda}, \qquad g = C\,e^{-\lambda} + D\,e^{\lambda}, \tag{3.1}$$

where the coefficients A, B, C, D are functions only of q. In detail the constraint (2.12) requires

$$(AD)_q = 2(e^q - e^{-q}), \qquad (BC)_q = 2(e^q - e^{-q}), \qquad A_q C = A C_q, \qquad B_q D = B D_q,$$

and hence

$$C = \alpha A, \qquad D = \alpha B, \qquad \alpha AB = 2(e^q + e^{-q}) + 2\gamma,$$

where α and γ are constants. Since λ can be shifted by a function of q without causing an essential change, there is a family of equivalent solutions to these constraints and it is a matter of convenience which choice is most suitable. For future purposes, it also turns out to be useful to define

$$\gamma = (e^{2\tau} + e^{-2\tau}).$$

A representative choice for f and g that will be used below is

$$f = \frac{1}{\sigma}(2\,\mathrm{e}^{p-\lambda} + \mathrm{e}^{-p+\lambda}(\mathrm{e}^{q} + \mathrm{e}^{-q} + \gamma)),$$

$$g = \sigma(\mathrm{e}^{\lambda}(\mathrm{e}^{q} + \mathrm{e}^{-q} + \gamma) + 2\,\mathrm{e}^{-\lambda}). \tag{3.2}$$

Using these, the defect conditions (2.2) can be rewritten in terms of p, q and λ as follows:

$$p_x - p_t + 2\lambda_t = -\frac{\sigma}{2}\mathrm{e}^{\lambda}(\mathrm{e}^{q} - \mathrm{e}^{-q}) - \frac{1}{2\sigma}\mathrm{e}^{-p+\lambda}(\mathrm{e}^{q} - \mathrm{e}^{-q}),$$

$$q_x - q_t = -\frac{\sigma}{2}(\mathrm{e}^{\lambda}(\mathrm{e}^{q} + \mathrm{e}^{-q} + \gamma) - 2\,\mathrm{e}^{-\lambda}), \tag{3.3}$$

$$q_x + q_t = \frac{1}{2\sigma}(\mathrm{e}^{-p+\lambda}(\mathrm{e}^{q} + \mathrm{e}^{-q} + \gamma) - 2\,\mathrm{e}^{p-\lambda}).$$

For the sinh-Gordon model, the static solution in the bulk is $u = v = 0$ and this satisfies the defect conditions (3.3) provided

$$\mathrm{e}^{2\lambda} = \frac{1}{2\cosh^2 \tau}.$$

On the other hand, purely imaginary solutions to the sinh-Gordon model are the solutions to the sine-Gordon model, the least energy static solutions in the bulk correspond to $u = 2\pi i a$ and $v = 2\pi i b$ where a and b are integers, and the defect conditions permit $a \neq b$ provided λ is chosen suitably. In fact, the conditions imply

$$\mathrm{e}^{2\lambda} = \begin{cases} 1/2\cosh^2 \tau & \text{if} \quad a - b \text{ is even} \\ 1/2\sinh^2 \tau & \text{if} \quad a - b \text{ is odd}. \end{cases} \tag{3.4}$$

3.2. The Liouville equation

The Liouville field theory fits into the same scheme by truncating the previous choices for f and g in the sinh/sine-Gordon model found in (3.2). Thus, for example,

$$f = 2\,\mathrm{e}^{p-\lambda}$$

$$g = \mathrm{e}^{\lambda}(\mathrm{e}^{q} + \mathrm{e}^{-q} + \gamma), \tag{3.5}$$

is an adequate choice since

$$\tfrac{1}{2}(f_q g_\lambda - f_\lambda g_q) = \mathrm{e}^{p+q} - \mathrm{e}^{p-q}.$$

In this case, there is no place for an arbitrary parameter to correspond to σ since any such could be removed by a translation of λ. On the other hand, the parameter γ can be chosen freely. Further, dropping one or the other of the exponential pieces e^{q} (or e^{-q}) in g leads to a defect that couples the Liouville model for u (or v) to free massless field theory for v (or u).

3.3. The Tzitzéica equation

For the Tzitzéica model the bulk potentials are

$$U = \mathrm{e}^{2p+2q} + 2\,\mathrm{e}^{-p-q} = \mathrm{e}^{2u} + 2\,\mathrm{e}^{-u}, \qquad V = \mathrm{e}^{2p-2q} + 2\,\mathrm{e}^{-p+q} = \mathrm{e}^{2v} + 2\,\mathrm{e}^{-v},$$

and the most general ansatz is

$$f = A\,\mathrm{e}^{2p-2\lambda} + B\,\mathrm{e}^{-p+\lambda}, \qquad g = C\,\mathrm{e}^{2\lambda} + D\,\mathrm{e}^{-\lambda}, \tag{3.6}$$

with the coefficients A, B, C, D being functions only of q. The constraints following from (2.12) are

$$A_q D = 2AD_q, \qquad 2B_q C = BC_q, \qquad (AC)_q = (\mathrm{e}^{2q} - \mathrm{e}^{-2q}), \qquad (BD)_q = 4(\mathrm{e}^{q} - \mathrm{e}^{-q}),$$

for which the general solution is

$$BD = 4(e^q - e^{-q}), \qquad A = \alpha D^2, \qquad C = \frac{B^2}{32\alpha}.$$

It is always possible to shift λ by a function of q and, for example, A (and therefore D) can be chosen to be constants. Using a further shift one of these constants may be removed and a convenient expression for the most general solution up to these translations of λ is

$$\begin{aligned}
f &= \frac{1}{\sigma}(e^{2p-2\lambda} + e^{-p+\lambda}(e^q + e^{-q})), \\
g &= \frac{\sigma}{2}(8\,e^{-\lambda} + e^{2\lambda}(e^q + e^{-q})^2).
\end{aligned} \tag{3.7}$$

This contains one free parameter σ.

3.4. Massive free fields

It is also instructive to consider the case where the fields to either side of the defect are free (and massive with mass parameter m). In this situation, similar considerations lead to

$$\begin{aligned}
f &= m\left(\frac{(p-\lambda)^2}{\beta} + \alpha q^2\right), \\
g &= m\left(\frac{\lambda^2}{\alpha} + \beta q^2\right),
\end{aligned} \tag{3.8}$$

where α and β are undetermined parameters.

One question is whether both of these parameters are effective after λ is eliminated (or, equivalently, integrated out in a functional integral). After some algebra, the result for the defect part of the Lagrangian (after removing a total time derivative) is as follows:

$$\mathcal{L}_D = \delta(x)\left[\frac{4\alpha\beta}{m(\alpha+\beta)}\,q_t^2 - \frac{1}{2}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)(uv_t - vu_t) - m\left(\frac{p^2}{\alpha+\beta} + (\alpha+\beta)q^2\right)\right]. \tag{3.9}$$

This still depends upon two parameters, yet in an interesting manner. For example, the limit $\alpha \to 0$ gives the free field type I defect considered in an earlier article [1], as does the limit $\beta \to 0$, apart from an inessential sign change in the term linear in time derivatives. From this observation it is clear that the new framework does indeed engender an alternative type of defect to those considered previously. However, it is not straightforward to eliminate λ in the other, nonlinear, examples.

The expressions for f and g in the sinh/sine-Gordon model given in (3.2) also contain two free parameters and it is to be expected that these survive in the quadratic limit regarded as an expansion about a classical constant configuration. One way to facilitate the limit is to put $\sigma = e^\eta$, and note an alternative but quite symmetrical expression for $\mathcal{D}$:

$$\mathcal{D} = 4\sqrt{2}\left(e^{-\lambda+p/2}\cosh\frac{p-2\eta}{2}\cosh\frac{q+2\tau}{2} + e^{\lambda-p/2}\cosh\frac{p+2\eta}{2}\cosh\frac{q-2\tau}{2}\right),$$

which may be expanded about the point $p = q = \lambda = 0$. After shifting

$$\lambda \to \lambda + \frac{q\tanh\tau}{2},$$

the quadratic form is diagonal and resembles (3.8); putting $m = \sqrt{2}$, α and β are given by

$$\alpha = \frac{\sigma}{2\cosh\tau}, \qquad \beta = \frac{1}{2\sigma\cosh\tau}.$$

These parameters lie on the set of curves $\alpha\beta = 1/4\cosh^2\tau$. On the other hand, the quadratic limit of expression (3.7) giving the functions f and g for the Tzitzéica equation is a particular one parameter set within the general two parameter family. Thus, for the Tzitzéica equation ($m = \sqrt{6}$) one finds

$$\alpha = \frac{1}{\sqrt{6}\sigma}, \qquad \beta = \frac{2\sigma}{\sqrt{6}}, \tag{3.10}$$

corresponding to points on the curve $\alpha\beta = \frac{1}{3}$.

If a plane travelling wave,

$$u = e^{-i\omega t}(e^{ikx} + R\,e^{-ikx}), \qquad v = e^{-i\omega t}T\,e^{ikx}, \qquad \omega = m\cosh\theta, \qquad k = m\sinh\theta,$$

encounters a defect with the potential (3.9) then there is no reflection ($R = 0$), and the transmission factor T is given by

$$T = \frac{i(\alpha\,e^\theta - \beta\,e^{-\theta}) + 1}{i(\alpha\,e^\theta - \beta\,e^{-\theta}) - 1}. \tag{3.11}$$

One difference from the previously considered cases (with $\alpha = 0$ or $\beta = 0$) is the possibility of a 'bound state' when $\alpha = \beta$, for example of the form

$$u = u_0\cos\omega t\,e^{m\zeta x}, \qquad x < 0; \qquad v = 0, \qquad x > 0, \qquad \zeta = -1/2\alpha,$$

with the constraint $\alpha < -1/2$. The contributions to the energy of this solution from the bulk and defect exactly cancel, though both are time-dependent, leading to a zero energy excitation degenerate with the constant 'vacuum' (in which all fields are zero everywhere).

Since the present scheme can accommodate all the known single field integrable Toda systems one might be optimistic that a generalization of the scheme will encompass all Toda models, conformal or affine, irrespective of the choice of root data. At this time, however, this generalization, if it exists, is not known.

4. A single soliton passing a defect

So far, nothing has been said about integrability. Nevertheless, this new class of defect is thought to be integrable on the basis of some indirect evidence. For example, if this is the case, at the very least single solitons for both the sine-Gordon model and the complex Tzitzéica model are expected to pass safely through a defect suffering at most a delay. In this section, the behaviour of single soliton solutions for these two models will be explored. In addition, in appendix A an energy-like spin-3 charge for the sine-Gordon model is calculated and found to be conserved on using the defect conditions (3.3). Ideally, a Lax pair formulation is needed to generalize the ideas presented in [1].

4.1. The sine-Gordon soliton

In the previous section, the sinh/sine-Gordon model were considered together but solitons are real solutions of sine-Gordon or purely imaginary solutions of the sinh-Gordon equation. For ease of notation and compatibility with earlier sections, the fields u and v will be pure imaginary. Then the defect conditions (3.3) will determine how a soliton scatters with the defect. The defect parameters will be taken to be real.

In a situation where the initial defect has either no discontinuity, or a discontinuity proportional to 4π, a single soliton solution can be written as follows:

$$e^{u/2} = \frac{1+E}{1-E}, \qquad E = e^{ax+bt+c}, \qquad a = \sqrt{2}\cosh\theta,$$

$$b = -\sqrt{2}\sinh\theta, \qquad e^{v/2} = \frac{1+zE}{1-zE},$$

where z represents the delay, the rapidity $\theta > 0$ indicates a soliton travelling from left to right along the x-axis and e^c is purely imaginary. Replacing $E \to -E$ (or equivalently shifting $c \to c + i\pi$) provides an expression for an anti-soliton. The final pair of defect conditions (3.3) do not involve λ_t and can be used to obtain two expressions for λ,

$$e^{\lambda} = -\frac{2}{\sigma} \frac{\sigma^2(q_x + q_t) + e^p(q_x - q_t)}{(e^p - e^{-p})(e^q + e^{-q} + \gamma)} \tag{4.1}$$

$$e^{-\lambda} = -\frac{1}{\sigma} \frac{\sigma^2(q_x + q_t) + e^{-p}(q_x - q_t)}{(e^p - e^{-p})}. \tag{4.2}$$

These two expressions must be consistent and will determine both z and λ. In fact, there will be two possible choices for z corresponding to the $i\pi$ ambiguity in the possible static solutions for λ given by (3.4). Explicitly, the two possibilities for the delay are given by $z = z_1$ or $z = z_2$, where

$$z_1 = \tanh\left(\frac{\theta - \eta + \tau}{2}\right) \tanh\left(\frac{\theta - \eta - \tau}{2}\right), \qquad z_2 = 1/z_1, \qquad \sigma = e^{\eta}. \tag{4.3}$$

For $z = z_1$, the companion expression for λ is given by

$$e^{\lambda_1} = \frac{1}{\sqrt{2}\cosh\tau} \frac{(1 + E_0)(1 + zE_0)}{(1 + \rho E_0)(1 + \tilde{\rho}E_0)},$$
$$\rho = \tanh\left(\frac{\theta - \eta + \tau}{2}\right), \tag{4.4}$$
$$\tilde{\rho} = \tanh\left(\frac{\theta - \eta - \tau}{2}\right),$$

where $E_0 = E(0, t)$, and there is a similar expression for λ_2.

Interestingly, expression (4.3) indicates that the delay is identical to a delay that would be experienced by a soliton passing through two defects of type I (see for example [5]) with parameters $\eta \pm \tau$. Because E_0 is purely imaginary expression (4.4) for λ_1 indicates that λ_1 is complex and nowhere singular as a function of real t. In order to decide which of the two solutions should be chosen the starting value of λ (that is, the value λ has when the soliton is far away but approaching the defect) needs to be specified—effectively, the defect has two states associated with it even when the static field configurations to either side of it are $u = 0$ and $v = 0$. The modulus of e^{λ_1}, with $0 < |z_1| \leqslant 1$, grows to a maximum at $E_0^4 = z_1^{-2}$ then falls to its initial value. On the other hand, the phase of e^{λ_1} is more interesting since it is the product of four terms, each having a single soliton (or anti-soliton) form (but is a function only of time):

$$e^{2i\mathrm{Im}\lambda_1} = \left(\frac{1 + E_0}{1 - E_0}\right)\left(\frac{1 + z_1 E_0}{1 - z_1 E_0}\right)\left(\frac{1 - \rho E_0}{1 + \rho E_0}\right)\left(\frac{1 - \tilde{\rho}E_0}{1 + \tilde{\rho}E_0}\right). \tag{4.5}$$

The first factor (provided E_0, which is pure imaginary, has a positive imaginary part) has a phase whose angle monotonically decreases by π as t runs over its range $(-\infty, \infty)$. On the other hand, if the imaginary part of E_0 had been negative, the phase angle would have increased by π. So, the total effect of the four terms will be either zero (if not more than one of ρ or $\tilde{\rho}$ is negative), or -4π (if both ρ and $\tilde{\rho}$ are negative). Thus the phase angle of e^{λ_1} either shifts by 0 or -2π. The case where the imaginary part of λ_1 shifts by -2π is quite interesting. There, the ingoing soliton emerges as a soliton but only after flipping to an anti-soliton and back again, in a virtual sense, since that is what would have happened had the soliton passed two separated defects with the chosen parameters. In other words, keeping track of λ distinguishes the two

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possible cases ($z_1 > 0$) where a soliton emerges as a soliton. In the other two cases (one of ρ or $\tilde{\rho}$ is negative), the soliton emerges as an antisoliton.

As was the case with type I defects, and as indicated above, the delay z can indicate a change in the character of the soliton as it passes (if $\eta - \tau < \theta < \eta + \tau$, then $z_1 < 0$, and an approaching soliton will emerge as an anti-soliton), or the soliton may be absorbed (if $\theta = \eta - \tau$ or $\theta = \eta + \tau$, meaning $z_1 = 0$). In the latter case, the expression for λ interpolates 'even' and 'odd' static solutions given by (3.4), as it should since the defect stores the topological charge (and the energy–momentum and other charges) transported by the soliton. The limit $\tau \to 0$ is interesting because in that limit the defect (at least as far as the scattering property is concerned) is behaving like another soliton of rapidity η. This lends a little more credibility to the idea (already mentioned in [5]) that a pair of defects with the same parameter behaves like a soliton. These results are very suggestive of the idea that at least for the sine-Gordon model the type II defects are 'squeezed', or 'fused', pairs of type I defects.

Finally, it is not difficult to check directly that the first of the three defect conditions (3.3) is satisfied by the soliton solution without any further constraints on λ or z.

A question that will not be addressed here is how the type II defect should be described by a transmission matrix in the quantum sine-Gordon field theory. Presumably, a generalization of the Konik–LeClair transmission matrix (see [5]) will need to be found and this will be postponed for a future investigation.

4.2. The Tzitzéica equation

The solitons for the Tzitzéica equation can be analysed similarly although in this case the soliton is complex (though its energy and momentum are real). Using the same conventions as before with the potential associated with the choice (3.7), the defect conditions are

$$
\begin{aligned}
p_x - p_t + 2\lambda_t &= -\frac{\sigma}{2}\, e^{2\lambda}(e^{2q} - e^{-2q}) - \frac{1}{2\sigma}\, e^{-p+\lambda}(e^q - e^{-q}), \\
q_x - q_t &= -\frac{\sigma}{2}(e^{2\lambda}(e^q + e^{-q})^2 - 4\, e^{-\lambda}), \\
q_x + q_t &= \frac{1}{2\sigma}(e^{-p+\lambda}(e^q + e^{-q}) - 2\, e^{2p-2\lambda}).
\end{aligned}
\tag{4.6}
$$

Single soliton solutions in the bulk are given by the expressions [20–22]

$$
e^u = \frac{(1 + E)^2}{(1 - 4E + e^2)}, \qquad e^v = \frac{(1 + zE)^2}{(1 - 4zE + z^2\, e^2)},
$$

with

$$
E = e^{ax+bt+c}, \qquad a = \sqrt{6}\cosh\theta, \qquad b = -\sqrt{6}\sinh\theta,
$$

where z represents the delay of the outgoing soliton. The constant e^c is chosen so that the expressions for u and v are nonsingular for all real choices of t and x. The last two of the defect conditions (4.6) can be regarded as a pair of cubic equations for $\Lambda \equiv e^\lambda$ of the form

$$
\alpha_1 \Lambda^3 + \beta_1 \Lambda^2 + \gamma_1 = 0, \qquad \alpha_2 \Lambda^3 + \beta_2 \Lambda + \gamma_2 = 0,
\tag{4.7}
$$

where the coefficients depend upon p, σ, q and the derivatives of q. Together, these may be solved to give

$$
\Lambda = \frac{\alpha_1 \beta_2^2 \gamma_1 + \beta_1 \gamma_2 (\alpha_1 \gamma_2 - \alpha_2 \gamma_1)}{\alpha_2 \beta_1 \beta_2 \gamma_1 - (\alpha_1 \gamma_2 - \alpha_2 \gamma_1)^2}, \qquad \frac{1}{\Lambda} = \frac{\alpha_2 \beta_1^2 \gamma_2 + \alpha_1 \beta_2 (\alpha_1 \gamma_2 - \alpha_2 \gamma_1)}{\alpha_2 \beta_1 \beta_2 \gamma_1 - (\alpha_1 \gamma_2 - \alpha_2 \gamma_1)^2}.
\tag{4.8}
$$

Demanding these two expressions are compatible and inserting the soliton solutions reveals, after some algebra, three possibilities for z:

$$z_1 = \frac{(e^{-\theta+\eta} + e^{i\pi/6})(e^{-\theta+\eta} + e^{-i\pi/6})}{(e^{-\theta+\eta} - e^{i\pi/6})(e^{-\theta+\eta} - e^{-i\pi/6})}, \qquad e^{\eta} = \sqrt{2}\sigma$$

$$z_2 = \bar{z}_3 = \frac{(e^{-\theta+\eta} - e^{i\pi/6})(e^{-\theta+\eta} + e^{-i\pi/2})}{(e^{-\theta+\eta} + e^{i\pi/6})(e^{-\theta+\eta} - e^{-i\pi/2})}.$$

(4.9)

These may also be rewritten more suggestively

$$z_1 = \coth\left(\frac{\theta - \eta}{2} - \frac{i\pi}{12}\right)\coth\left(\frac{\theta - \eta}{2} + \frac{i\pi}{12}\right),$$

(4.10)

$$z_2 = \bar{z}_3 = \coth\left(\frac{\theta - \eta}{2} + \frac{i\pi}{4}\right)\tanh\left(\frac{\theta - \eta}{2} - \frac{i\pi}{12}\right),$$

(4.11)

and $z_1 z_2 z_3 = 1$. Finally, as examples, for the two cases $z = z_1$ or $z = z_2$ expressions for the field λ are

$$e^{\lambda_1} = \frac{(1 + E_0)(1 + z_1 E_0)}{\left(1 + 2\rho_1 E_0 + z_1 E_0^2\right)}, \qquad \rho_1 = \frac{(e^{-\theta+\eta} - \sqrt{2})(e^{-\theta+\eta} + \sqrt{2})}{(e^{-\theta+\eta} - e^{i\pi/6})(e^{-\theta+\eta} - e^{-i\pi/6})},$$

(4.12)

$$e^{\lambda_2} = \frac{(1 + E_0)(1 + z_2 E_0)}{\left(1 + 2\rho_2 E_0 + z_2 E_0^2\right)} e^{-2i\pi/3}, \qquad \rho_2 = \frac{(e^{-\theta+\eta} - \sqrt{2}\,e^{i\pi/3})(e^{-\theta+\eta} + \sqrt{2}e^{i\pi/3})}{(e^{-\theta+\eta} + e^{i\pi/6})(e^{-\theta+\eta} - e^{-i\pi/2})}.$$

(4.13)

For $z = z_3$ the corresponding formulae are the complex conjugates of the expressions in (4.13). The possible asymptotic values of u and v for soliton solutions are $u = 2\pi i a$, $v = 2\pi i b$, and the corresponding asymptotic values of λ required by the defect conditions are $\lambda = 2\pi i c$ or $\lambda = \pm 2\pi i/3 + 2\pi i c$ with a, b, c integers. Formulae (4.12) and (4.13) for λ provide examples of this. Once again, as was found to be the case for the sine-Gordon model, part of the specification of the defect must be the initial choice of λ (essentially, for the soliton, one of three).

5. Defects and Bäcklund transformations

In [1] it was pointed out using several examples that integrable defect conditions for type I defects coincide with Bäcklund transformations 'frozen' at the defect location. This impression was strongly reinforced by subsequent analysis of the $a_n^{(1)}$ affine Toda models [2, 6]. However, it was also found that while the $a_2^{(2)}$ Toda model has Bäcklund transformations these cannot be used directly to construct integrable defects within the type I scheme. At first sight this seemed puzzling and the purpose of this section is to show how a 'folding' procedure [3, 23] may be used to obtain a Bäcklund transformation for the Tzitzéica model, making use of two similar, yet different, sets of defect conditions obtained in [2] for the $a_2^{(1)}$ Toda model.

First a little background is necessary. The equation of motion for an $a_2^{(1)}$ Toda field u is

$$\partial^2 u = -2\sum_{j=0}^{2} \alpha_j\, e^{\alpha_j \cdot u},$$

(5.1)

where, with respect to a basis of orthonormal vectors $\{e_0, e_1, e_2\}$ in a three-dimensional Euclidean space, the $a_2^{(1)}$ roots are

$$\alpha_1 = e_1 - e_2, \qquad \alpha_2 = e_2 - e_0, \qquad \alpha_0 = e_0 - e_1.$$

(5.2)

The projections of the field u onto the orthonormal basis are u_0, u_1, u_2 satisfying the constraint $u_0 + u_1 + u_2 = 0$. From (5.1) it follows that the corresponding equations for the projections read

$$\partial^2 u_j = -2(e^{u_j - u_{j+1}} - e^{-u_j + u_{j-1}}), \qquad j = 0, 1, 2, \tag{5.3}$$

where the subscripts on the right-hand side are to be understood modulo 3. Then, the folding procedure consists of setting one of the fields to zero, for instance $u_2 = 0$ (i.e. $u_1 = -u_0$), to obtain the Tzitzéica equation of motion with the same normalizations as had been assumed when writing down the Tzitzéica potential in section 4.2. Note, the alternative choices $u_0 = 0$ or $u_1 = 0$ would lead to the same conclusion. The defect conditions that must hold at the defect ($x = x_0$) when sewing together two $a_2^{(1)}$ Toda fields u and λ are

$$\partial_x u - A\partial_t u - B\partial_t \lambda + \mathcal{D}_u = 0, \qquad \partial_x \lambda - B^T \partial_t u + A\partial_t \lambda - \mathcal{D}_\lambda = 0, \qquad B = (1 - A), \tag{5.4}$$

with

$$\mathcal{D} = \sqrt{2} \sum_{j=0}^{2} \left(\sigma\, e^{\alpha_j \cdot (B^T u + B\lambda)/2} + \frac{1}{\sigma}\, e^{\alpha_j \cdot B(u-\lambda)/2} \right), \qquad B = 2\sum_{a=0}^{2} w_a(w_a - w_{a+1})^T, \tag{5.5}$$

where w_1, w_2 ($w_3 \equiv w_0 = 0$) are the fundamental highest weights of the Lie algebra $a_2^{(1)}$. By using similar notation as in (5.3) for the two fields u and λ and light-cone coordinates $x_\pm = (t \pm x)/2$, the full set of defect conditions (5.4) read

$$\partial_+(u_1 - u_2) - \partial_+(\lambda_1 - \lambda_2) = \sqrt{2}\,\sigma(e^{u_0 - \lambda_1} - 2\,e^{u_1 - \lambda_2} + e^{u_2 - \lambda_0}),$$
$$\partial_+(u_2 - u_0) - \partial_+(\lambda_2 - \lambda_0) = \sqrt{2}\,\sigma(e^{u_1 - \lambda_2} - 2\,e^{u_2 - \lambda_0} + e^{u_0 - \lambda_1}),$$
$$\partial_+(u_0 - u_1) - \partial_+(\lambda_0 - \lambda_1) = \sqrt{2}\,\sigma(e^{u_2 - \lambda_0} - 2\,e^{u_0 - \lambda_1} + e^{u_1 - \lambda_2}),$$
$$\partial_-(u_1 - u_2) - \partial_-(\lambda_2 - \lambda_0) = \sqrt{2}\,\sigma^{-1}(2\,e^{-u_2 + \lambda_2} - e^{-u_1 + \lambda_1} - e^{-u_0 + \lambda_0}), \tag{5.6}$$
$$\partial_-(u_2 - u_0) - \partial_-(\lambda_0 - \lambda_1) = \sqrt{2}\,\sigma^{-1}(2\,e^{-u_0 + \lambda_0} - e^{-u_2 + \lambda_2} - e^{-u_1 + \lambda_1}),$$
$$\partial_-(u_0 - u_1) - \partial_-(\lambda_1 - \lambda_2) = \sqrt{2}\,\sigma^{-1}(2\,e^{-u_1 + \lambda_1} - e^{-u_0 + \lambda_0} - e^{-u_2 + \lambda_2}).$$

In the bulk, expression (5.6) would be the Bäcklund transformation discovered by Fordy and Gibbons [12].

Unfortunately, the folding procedure cannot be applied directly to the defect conditions because they are simply incompatible with folding. This fact can be expressed heuristically by noting that the defect conditions (5.6) do not treat solitons and antisolitons identically (a feature already pointed out in [2, 6] and expected since solitons and anti-solitons are associated with different representations of the a_2 algebra), because each type of soliton experiences a different delay on transmission through the defect. The soliton solution of the Tzitzéica model can be thought of as a particular soliton–antisoliton solution of the $a_2^{(1)}$ affine Toda model, and, since the components of a multi-soliton are delayed independently by the defect, its components will be treated differently by (5.6). Therefore, the Tzitzéica soliton cannot survive intact. A remedy is provided by observing that an alternative defect setting is available if the matrix B in (5.5) is replaced by its transpose. The resulting set of defect conditions describes a system, which is still integrable yet interchanges the delays experienced by a soliton or antisoliton when compared with the previous case. This suggests that two different types of defect, one built using B (at $x = x_0$) and the other with B^T (at $x = x_1$), then 'squeezed' together ($x_1 \to x_0$), might allow the folding procedure to be applied successfully. The second set of defect conditions matches two $a_2^{(1)}$ fields λ and v and would be written in a similar manner to (5.6) but using B^T instead of B.

Since the incoming (u) and outgoing (v) solitons are required to satisfy the Tzitzéica equation of motion, the projections u_2, v_2 can be set equal to zero. Consequently, the field λ is forced to satisfy the following constraint (at $x = x_0$):

$$2\,e^{-\lambda_0} = e^{-p_0+\lambda_2}\,(e^{q_0} + e^{-q_0}), \qquad p_0 = \frac{u_0 + v_0}{2}, \qquad q_0 = \frac{u_0 - v_0}{2}. \tag{5.7}$$

Setting $u_0 \equiv u$, $v_0 \equiv v$, $\lambda_2 \equiv -\lambda$ and sending $\sigma \to 1/(\sqrt{2}\,\sigma)$ the two sets of defect conditions lead to

$$\partial_-(p - \lambda) = \frac{\sigma}{2}\,e^{2\lambda}(e^{2q} - e^{-2q}) \tag{5.8}$$

$$\partial_+\lambda = -\frac{1}{2\sigma}e^{-p+\lambda}(e^{q} - e^{-q}), \tag{5.9}$$

$$\partial_-q = \frac{\sigma}{2}\,(e^{2\lambda}(e^{q} + e^{-q})^2 - 4\,e^{-\lambda}), \tag{5.10}$$

$$\partial_+q = \frac{1}{2\sigma}(e^{-p+\lambda}(e^{q} + e^{-q}) - 2\,e^{2p-2\lambda}). \tag{5.11}$$

If, instead of being 'frozen' at $x = x_0$, equations (5.8)–(5.11) were required to hold in the bulk, they do, in fact, represent a Bäcklund transformation for the Tzitzéica equation. This can be seen by cross-differentiating expressions (5.10) and (5.11) to eliminate λ, to find that if the field u satisfies the Tzitzéica equation then the field v also satisfies it. Also, cross-differentiating expressions (5.8) and (5.9) an equation of motion satisfied by the field λ emerges

$$\partial^2\lambda = -(e^{q} + e^{-q})\,e^{\lambda-p}(e^{2\lambda} - e^{-\lambda}). \tag{5.12}$$

Inevitably, this depends on the fields u and v. The Bäcklund transformation (5.8)–(5.11) seems not to have been reported elsewhere in the literature [14–16].

On the other hand, since equations (5.8)–(5.11) are supposed to hold only at $x = x_0$, and because the quantity λ is confined at $x = x_0$ and depends only on t, the sum of the pair (5.8) and (5.9), together with (5.10) and (5.11) are precisely the three defect conditions (4.6). Hence, for the type II defect, the number of defect conditions following from the Lagrangian (2.1) is one less than the number of equations specifying the Bäcklund transformation described above. This is quite different to the previous situation where the Lagrangian description of a type I defect led directly to the frozen Bäcklund transformation (and hence to the Bäcklund transformation itself).

Clearly, using the same idea, the defect conditions (3.3) can be augmented to obtain an alternate Bäcklund transformation for the sine-Gordon model that depends on two parameters:

$$\begin{aligned}
\partial_-(p - \lambda) &= \frac{\sigma}{2}\,e^{\lambda}(e^{q} - e^{-q}) \\
\partial_+\lambda &= -\frac{1}{2\sigma}e^{-p+\lambda}(e^{q} - e^{-q}), \\
\partial_-q &= \frac{\sigma}{2}\,(e^{\lambda}(e^{q} + e^{-q} + \gamma) - 2\,e^{-\lambda}), \\
\partial_+q &= \frac{1}{2\sigma}(e^{-p+\lambda}(e^{q} + e^{-q} + \gamma) - 2\,e^{(p-\lambda)}).
\end{aligned} \tag{5.13}$$

From these relations, in a similar manner as previously, the equations of motion for the sine-Gordon fields u and v are recovered and the field λ satisfies

$$\partial^2\lambda = -\tfrac{1}{4}\,e^{-p}\,(4\,e^{2\lambda} - (e^{q} + e^{-q})(2 - \gamma\,e^{2\lambda})). \tag{5.14}$$

The fact that there appear to be generalizations of the defect conditions, which are only indirectly related to Bäcklund transformations, and yet likely to be integrable, generates a sense of optimism that the framework will generalize to encompass all affine Toda models.

6. Defects as Hamiltonian constraints

So far, properties of defects, and the relationship of the defect conditions to the conservation of a suitably defined momentum, have been derived from first principles from a Lagrangian starting point. It is interesting to ask if the framework can be formulated within a Hamiltonian setting. In this section this will be attempted, at least at a formal level, by explaining the main ideas, albeit sketchily. The setup demonstrates explicitly that the presence of a defect reduces the independent degrees of freedom of the system in the sense of providing defect conditions that can be regarded as a set of constraints on the fields u and v (for type I defects), or u, v and λ (for type II defects). This fact is highlighted by the emergence of second class constraints in the Hamiltonian (for a detailed description of these, see for example [24]).

The discussion can begin by considering a system with a type I defect. In this case, the starting point is the following Lagrangian density:

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(\frac{u\,v_t - v\,u_t}{2} - \mathcal{D}(u, v)\right), \tag{6.1}$$

with

$$\mathcal{L}_u = \tfrac{1}{2}\partial_\mu u\,\partial^\mu u - U(u), \qquad \mathcal{L}_v = \tfrac{1}{2}\partial_\mu v\,\partial^\mu v - V(v). \tag{6.2}$$

According to the usual definitions, and treating the theta and delta functions formally, the canonical momenta conjugate to the fields u and v are

$$\pi_u = \frac{\partial \mathcal{L}}{\partial u_t} = \theta(-x)\,u_t - \delta(x)\,\frac{v}{2},$$
$$\pi_v = \frac{\partial \mathcal{L}}{\partial v_t} = \theta(x)\,v_t + \delta(x)\,\frac{u}{2}. \tag{6.3}$$

By comparison with what happens within each half line, $x < 0$ or $x > 0$, the canonical momenta are not well defined at the defect location. In other words, at $x = 0$ it is not possible to write the time derivatives of the fields (Lagrangian variables) in terms of the canonical momenta (Hamiltonian variables). At $x = 0$ the canonical momenta are not independent, and the definitions (6.3) provide constraints amongst the canonical variables. These are

$$\chi_1 = \pi_u + \frac{v}{2} = 0, \qquad \chi_2 = \pi_v - \frac{u}{2} = 0, \tag{6.4}$$

these are primary constraints. The Hamiltonian is given by

$$H = \int_{-\infty}^{\infty} \mathrm{d}x\,\mathcal{H} \tag{6.5}$$

with

$$\mathcal{H} = \theta(-x)\left(\frac{\pi_u^2 + u_x^2}{2} + U\right) + \theta(x)\left(\frac{\pi_v^2 + v_x^2}{2} + V\right) + \delta(x)\left(\mathcal{D} + \mu_1\chi_1 + \mu_2\chi_2\right), \tag{6.6}$$

where μ_1 and μ_2 are functions of the fields u, v together with their momenta. They can be determined by using the fact that the constraints χ_1 and χ_2 must be preserved in time. In other words, the relations

$$\chi_{1\,t} = \{\chi_1, H\} = 0, \qquad \chi_{2\,t} = \{\chi_2, H\} = 0, \tag{6.7}$$

must hold. The Poisson bracket of two functionals $F = \int_{-\infty}^{\infty} \mathrm{d}x\,\mathcal{F}$ and $G = \int_{-\infty}^{\infty} \mathrm{d}x\,\mathcal{G}$ is defined formally as follows:

$$\{F, G\} = \int_{-\infty}^{\infty} \mathrm{d}x\left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta \pi_u} - \frac{\delta F}{\delta \pi_u}\frac{\delta G}{\delta u}\right) + \int_{-\infty}^{\infty} \mathrm{d}x\left(\frac{\delta F}{\delta v}\frac{\delta G}{\delta \pi_v} - \frac{\delta F}{\delta \pi_v}\frac{\delta G}{\delta v}\right). \tag{6.8}$$

Using this definition and the Hamiltonian (6.5), for which,

$$\frac{\delta H}{\delta \pi_u} \equiv u_t = \frac{\partial \mathcal{H}}{\partial \pi_u} = \theta(-x)\pi_u + \delta(x)\mu_1, \qquad \frac{\delta H}{\delta \pi_v} \equiv v_t = \frac{\partial \mathcal{H}}{\partial \pi_v} = \theta(x)\pi_v + \delta(x)\mu_2,$$

$$\frac{\delta H}{\delta u} \equiv -\pi_{u\,t} = \frac{\partial \mathcal{H}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{H}}{\partial u_x} = \theta(-x)(-u_{xx} + U') + \delta(x)\left(\mathcal{D}_u - \frac{\mu_2}{2} + u_x\right), \qquad (6.9)$$

$$\frac{\delta H}{\delta v} \equiv -\pi_{v\,t} = \frac{\partial \mathcal{H}}{\partial v} - \frac{\partial}{\partial x}\frac{\partial \mathcal{H}}{\partial v_x} = \theta(x)(-v_{xx} + V') + \delta(x)\left(\mathcal{D}_v + \frac{\mu_1}{2} - v_x\right),$$

where $U' = U_u$ and $V' = V_v$, the Poisson brackets (6.7) can be calculated. In consequence, (6.7) leads to explicit expressions for the functions μ_1 and μ_2, which are

$$\mu_1 = -\mathcal{D}_v + v_x \qquad \mu_2 = \mathcal{D}_u + u_x.$$

Assembling all these ingredients, the Hamiltonian density (6.6) becomes

$$\mathcal{H} = \theta(-x)\left(\frac{\pi_u^2 + u_x^2}{2} + U\right) + \theta(x)\left(\frac{\pi_v^2 + v_x^2}{2} + U\right)$$

$$+ \delta(x)\left[\left(\pi_u + \frac{v}{2}\right)(v_x - \mathcal{D}_v) + \left(\pi_v - \frac{u}{2}\right)(u_x + \mathcal{D}_u) + \mathcal{D}\right]. \qquad (6.10)$$

Expressions (6.9) are the canonical Hamilton equations and using the definitions of the canonical momenta they coincide with the defect conditions and equations of motion of the type I defect problem (the latter by performing a differentiation with respect to time).

In principle, the conservation of any charge can be verified by calculating its Poisson bracket with the Hamiltonian. For example, consider the total momentum of the system, which is defined by

$$P = \int_{-\infty}^{\infty} dx\, \mathcal{P} \quad \text{with} \quad \mathcal{P} = \theta(-x)\pi_u u_x + \theta(x)\pi_v v_x + \delta(x)\Omega(u,v). \qquad (6.11)$$

It is straightforward to calculate the time derivative of P using its Poisson bracket with the Hamiltonian to obtain

$$\dot{P} = \delta(x)\left[\frac{1}{2}\left(\mathcal{D}_u^2 - \mathcal{D}_v^2\right) - U + V + u_t(\Omega_u - \mathcal{D}_v) + v_t(\Omega_v - \mathcal{D}_u)\right] = 0. \qquad (6.12)$$

The final step follows from the facts that $\left(\mathcal{D}_u^2 - \mathcal{D}_v^2\right)/2 = (U - V)$, $\mathcal{D} = (f + g)$ and $\Omega = (f - g)$, with $f = f(p)$ and $g = g(q)$, as was described previously in [1].

It should be noted that the constraints (6.4) are second class. Hence, as mentioned at the beginning of this section, they indicate that not all degrees of freedom are independent. By definition, a constraint is first class if its Poisson brackets with all other constraints are zero—the constraints themselves can be imposed, if needed—-otherwise, it is second class. In the present case, it is straightforward to check that the Poisson brackets of the constraints are constant. In fact,

$$C_{ij} \equiv \{\chi_i, \chi_j\}, \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix C can be used to construct the Dirac brackets, the standard tool for dealing with second class constraints.

Next, consider the type II defect and suppose the Lagrangian density is given by (2.1). Then, there are three fields u, v and λ, whose canonical momenta are

$$\pi_u = \frac{\partial \mathcal{L}}{\partial u_t} = \theta(-x)\, u_t - \delta(x) \left(\frac{v}{2} - \lambda \right),$$

$$\pi_v = \frac{\partial \mathcal{L}}{\partial v_t} = \theta(x)\, v_t + \delta(x) \left(\frac{u}{2} - \lambda \right),$$

$$\pi_\lambda = \frac{\partial \mathcal{L}}{\partial \lambda_t} = -\delta(x)(u - v).$$

Consequently, the primary constraints are

$$\chi_1 = \pi_u + \frac{v}{2} - \lambda = 0, \qquad \chi_2 = \pi_v - \frac{u}{2} + \lambda = 0, \qquad \chi_3 = \pi_\lambda + (u - v) = 0, \qquad (6.13)$$

and the Hamiltonian density reads

$$\mathcal{H} = \theta(-x) \left(\frac{\pi_u^2 + u_x^2}{2} + U \right) + \theta(x) \left(\frac{\pi_v^2 + v_x^2}{2} + V \right) + \delta(x)(\mathcal{D} + \mu_1 \chi_1 + \mu_2 \chi_2 + \mu_3 \chi_3). \tag{6.14}$$

Since these constraints must be consistent with the evolution equations, their time derivative must vanish. By using the following Poisson bracket:

$$\{F, G\} = \int_{-\infty}^{\infty} dx \left(\frac{\delta F}{\delta u} \frac{\delta G}{\delta \pi_u} - \frac{\delta F}{\delta \pi_u} \frac{\delta G}{\delta u} \right) + \int_{-\infty}^{\infty} dx \left(\frac{\delta F}{\delta v} \frac{\delta G}{\delta \pi_v} - \frac{\delta F}{\delta \pi_v} \frac{\delta G}{\delta v} \right)$$
$$+ \left(\frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta \pi_\lambda} - \frac{\delta F}{\delta \pi_\lambda} \frac{\delta G}{\delta \lambda} \right)_{x=0}, \tag{6.15}$$

it is possible to verify that

$$\chi_{1\,t} = -\mathcal{D}_u - u_x + \mu_2 - 2\mu_3 = 0,$$
$$\chi_{2\,t} = -\mathcal{D}_v + v_x - \mu_1 + 2\mu_3 = 0,$$
$$\chi_{3\,t} = -\mathcal{D}_\lambda + 2(\mu_1 - \mu_2) = 0.$$

Unlike the previous case, this system of equations does not determine completely the functions μ_j. In fact, requiring the constraints to be preserved with time forces

$$\mu_1 = -\mathcal{D}_v + v_x + 2\mu_3, \qquad \mu_2 = \mathcal{D}_u + u_x + 2\mu_3 \tag{6.16}$$

$$(u - v)_x + \mathcal{D}_u + \mathcal{D}_v + \tfrac{1}{2}\mathcal{D}_\lambda = 0. \tag{6.17}$$

Expression (6.17) is a secondary constraint. However, it is not genuinely new since it coincides with an algebraic sum of some of the canonical Hamiltonian equations, as can be verified by using the following Hamiltonian density:

$$\mathcal{H} = \theta(-x) \left(\frac{\pi_u^2 + u_x^2}{2} + U \right) + \theta(x) \left(\frac{\pi_v^2 + v_x^2}{2} + V \right) + \delta(x)(\mathcal{D} + \mu_3(2\pi_u + 2\pi_v + \pi_\lambda))$$
$$+ \delta(x) \left[\left(\pi_u + \frac{v}{2} - \lambda \right)(v_x - \mathcal{D}_v) + \left(\pi_v - \frac{u}{2} + \lambda \right)(u_x + \mathcal{D}_u) \right]. \tag{6.18}$$

In fact,

$$0 = (\pi_\lambda + 2\pi_u + 2\pi_v)_t \equiv -(u - v)_x - \mathcal{D}_u - \mathcal{D}_v - \tfrac{1}{2}\mathcal{D}_\lambda,$$

which coincides with (6.17). As was shown in the previous case, all Hamilton equations can be obtained and they lead to the equations of motion and defect conditions (note that $\lambda_t \equiv \mu_3$). Finally, as mentioned before, the Poisson brackets (6.15) may be used to verify the

conservation of charges. For example, given the total momentum (6.11), it can be checked that

$$\dot{P} = \delta(x)\left[\frac{1}{2}\left(\mathcal{D}_u^2 - \mathcal{D}_v^2\right) - U + V + \lambda_t(\mathcal{D}_\lambda + 2\mathcal{D}_u + 2\mathcal{D}_v + \Omega_\lambda)\right.$$

$$\left. + u_t(\Omega_u - D_v) + v_t(\Omega_v - D_u)\right].$$

Since $D = (f + g)$ and $\Omega = (f - g)$, with $f = f(p - \lambda, q)$ and $g = g(\lambda, q)$, the above expression becomes

$$\dot{P} = \delta(x)\left[\frac{1}{2}\left(\mathcal{D}_u^2 - \mathcal{D}_v^2\right) - U + V + \frac{1}{2}f_q\mathcal{D}_\lambda\right] \equiv 0.$$

In summary, from the Hamiltonian density (6.18), it is possible to read off the final constraints, which are

$$\chi_1, \qquad \chi_2, \qquad \gamma_1 = 2\pi_u + 2\pi_v + \pi_\lambda,$$

where χ_1, χ_2 are second class, while γ_1 is first class. In fact, it can be checked that $\{\chi_1, \gamma_1\} = \{\chi_2, \gamma_1\} = \{\gamma_1, \gamma_1\} = 0$. The first class constraints are usually related to the presence of a gauge freedom. In the type II defect framework, the existence of a first class constraint indicates the freedom to translate the field λ by any function of q, as was pointed out in section 3.

7. Comments and conclusions

The main result of this paper has been to extend the framework within which an integrable defect may be described. The previous framework (referred to as type I in this paper) seemed fairly natural yet even for a single scalar field was unable to accommodate all possible relativistic integrable models because the Tzitzéica, or $a_2^{(2)}$ affine Toda, model was conspicuously absent. For multiple scalar fields the possible type I defects are restricted to the $a_n^{(1)}$ series of affine Toda models. In all cases, the type I defects are intimately related to Bäcklund transformations, in the sense that the conditions relating the fields on either side of an integrable defect take the form of a Bäcklund transformation frozen at the location of the defect. At first sight, this relationship seemed attractive since it provided a use for Bäcklund transformations that had not been noticed before. On the other hand, the Tzitzéica equation has several Bäcklund transformations associated with it and none of them emerged naturally from within the type I framework. Moreover, the integrability of the type I defects is intimately related to momentum conservation, in the sense that insisting there should be a total momentum including a contribution from the defect itself leads to restrictions that would be associated normally with the requirements of having higher spin conserved quantities. It is a curious situation: certain integrable systems (those with type I defects) can violate translational invariance yet preserve momentum. The question is: can this phenomenon be extended to other integrable systems by changing the framework? It appears the answer is yes, and one particular different framework (referred to as type II) is described in this paper. In fact, only a slight change appears to be necessary, the Tzitzéica model is incorporated, and the relationship with frozen Bäcklund transformations is modified. The trick is to introduce a new degree of freedom located on the defect and couple it in a minimal manner to the discontinuity across the defect. In the absence of a generalized Lax pair for the type II system, momentum conservation becomes a tool for identifying the possibilities, backed up by other less direct evidence. Turning the argument around and starting from the defect conditions allows an apparently new Bäcklund transformation to be established for the Tzitzéica equation.

The type II framework certainly contain all single field integrable systems of Toda type (or free fields) but it is not yet demonstrated these are the only possibilities. The latter appears reasonable since (2.12) is highly constraining but a complete proof of integrability needs to be found in order to be sure.

It is already known that the $a_n^{(1)}$ affine Toda models can support type I defects of several kinds and that defects are able to relate different a_n conformal Toda models to each other (thereby generalizing the relationship between the Liouville model and free fields [10]). However, other affine Toda models based on the root data of the b, c, d, e, f, g series of Lie algebras do not appear to fit in to the type I framework. This is surprising: in most respects, the affine Toda field theories at least in the bulk, have similar features, though it does appear from the literature that the $a_n^{(1)}$ series is special in having a Bäcklund transformation of a simple type. It remains to be seen if the type II framework can be adapted to all Toda models. The folding process cannot explain the apparent difficulties with the d, e series. However, once these are understood the folding process might be an essential part of the story for the remaining cases. For that reason it would be natural to examine the d, e series next.

At this stage it is worth outlining a possible direction for a generalization containing multi-component fields. Using the same notation as previously, taking as a starting point the defect contribution

$$\mathcal{L}_D = \delta(x) \left(q \cdot A q_t + 2\lambda \cdot q_t - \mathcal{D}(\lambda, q, p) \right), \tag{7.1}$$

where A is an antisymmetric matrix, then insisting on overall momentum conservation, leads to the following constraints on $\mathcal{D}$ and Ω:

$$\mathcal{D} = f(p - \lambda, q) + g(p + \lambda, q), \qquad \Omega = f(p - \lambda, q) - g(p + \lambda, q). \tag{7.2}$$

Further, the two functions f and g are constrained by a generalization of the Poisson bracket relation (2.12) that reads

$$\nabla_q f \cdot \nabla_\lambda g - \nabla_q g \cdot \nabla_\lambda f + \nabla_\lambda f \cdot A \nabla_\lambda g = U(u) - V(v). \tag{7.3}$$

Here A is the antisymmetric matrix occurring in (7.1) and U, V are the bulk potentials for the fields to either side of the defect. The left-hand side of (7.3) is a bona fide Poisson bracket since it is antisymmetric and satisfies the Jacobi relation, yet, as before, all dependence on λ must cancel out. This provides severe constraints on U and V, which will be explored elsewhere.

At the quantum level, it was demonstrated in [5, 6] that type I defects within the $a_n^{(1)}$ series are described by infinite-dimensional transmission matrices, which are determined up to a single parameter by a set of 'triangle relations' ensuring their compatibility with the bulk S-matrix. Moreover, arguments have been provided to demonstrate that the free parameter is essentially the same, though possibly renormalized, as the free parameter in the type I Lagrangian. Clearly, the next question concerns the transmission matrix in the context of type II defects. For the sine-Gordon model, the transmission matrix in this framework should depend on two independent parameters and there should be some evidence or influence of the confined field λ, at least recognizing the iπ ambiguity mentioned in section 4.1. At a quantum level, the Tzitzéica model contains a triplet of equal mass states, reflecting its origin in $a_2^{(1)}$ affine Toda field theory under the folding process, only two of which correspond to classical solitons, and its S-matrix is known [25]. It is to be hoped there will be a transmission matrix based on an ansatz that takes into account the mysterious role of λ (this time the ambiguity is threefold—see section 4.2).

Appendix. Energy-like spin-3 charge for the sine-Gordon model

In this appendix, it is shown that an energy-like spin-3 charge for the sine-Gordon model with a defect of type II is conserved. The bulk charge, which is not expected to be conserved in the presence of a defect, conveniently normalized, reads

$$
\mathcal{E}_3 = \int_{-\infty}^{0} \mathrm{d}x \left(\frac{u_t^4 + u_x^4}{4} + \frac{3}{2} u_x^2 u_t^2 + 4u_{tx}^2 + (u_{tt} + u_{xx})^2 + \left(u_t^2 + u_x^2\right) U'' \right)
$$
$$
+ \int_{0}^{\infty} \mathrm{d}x \left(\frac{v_t^4 + v_x^4}{4} + \frac{3}{2} v_x^2 v_t^2 + 4v_{tx}^2 + (v_{tt} + v_{xx})^2 + \left(v_t^2 + v_x^2\right) V'' \right),
$$

and its time derivative is

$$
\dot{\mathcal{E}}_3 = \left[\left(u_t u_x^3 + u_t^3 u_x\right) - \left(v_t v_x^3 + v_t^3 v_x\right) + 4(2u_{tt} + U')u_{tx} \right.
$$
$$
\left. - 4\left(2v_{tt} + V'\right)v_{tx} - 2(u_t u_x U'' - v_t v_x V'') \right]_{x=0}, \tag{A.1}
$$

where $U' = U_u$ and $V' = V_v$. This is not expected to be zero but the right-hand side may turn out to be the total time derivative of a functional $-\mathcal{D}_3$ that depends only on the defect variables p, q and λ. In that case, $\mathcal{E}_3 + \mathcal{D}_3$ will be conserved. Since expression (A.1) is calculated at $x = 0$, it is convenient to rewrite it by using the variables p and q. Then, using the defect conditions (2.2)–(2.4) with the functions f and g given by (3.2), expression (A.1) becomes a total time derivative

$$
\dot{\mathcal{E}}_3 = 4 \frac{\mathrm{d}}{\mathrm{d}t} \left(2(p_t - \lambda_t) q_t f_{\lambda q} - (p_t - \lambda_t)^2 f - q_t^2 (f+g)_{qq} - \lambda_t^2 g - 2q_t \lambda_t g_{\lambda q} \right)
$$
$$
+ 4 \frac{\mathrm{d}}{\mathrm{d}t} \left((p_t - \lambda_t)(U' - V') - \lambda_t(U' - V') - q_t(U' + V') \right) - \frac{\mathrm{d}}{\mathrm{d}t} \Omega_3(p, q, \lambda), \tag{A.2}
$$

(where again on the right-hand side all field quantities are evaluated at $x = 0$), with

$$
\frac{\partial \Omega_3}{\partial q} = 3 f_\lambda (U - V) - \frac{3}{4} f_q (f+g)_\lambda^2 + \frac{1}{4}(f+g)_q \left(3 f_\lambda^2 + (f+g)_q^2 - 12(U+V)\right),
$$
$$
\frac{\partial \Omega_3}{\partial p} = -3(f+g)_q (U - V) + \frac{3}{4} f_{qq}(f+g)_\lambda^2 - \frac{1}{4} f_\lambda \left(f_\lambda^2 + 3(f+g)_q^2 - 12(U+V)\right),
$$
$$
\frac{\partial \Omega_3}{\partial \lambda} = \frac{1}{4}(f+g)_\lambda \left(f_\lambda^2 + g_\lambda^2 - f_\lambda g_\lambda - 3(f+g)_\lambda (f+g)_{qq} + 3(f+g)_q^2 - 12(U+V)\right).
$$

Formula (A.2) has been obtained by making use of the following properties of the defect potential for the sine-Gordon model

$$
\begin{aligned}
&f_p = -f_\lambda, & &f_{\lambda\lambda} = f, & &g_{\lambda\lambda} = g, \\
&f_{qqq} = f_q, & &g_{qqq} = g_q, & &f_{\lambda q} = f_q, & &g_{\lambda q} = g_q.
\end{aligned} \tag{A.3}
$$

Finally, it has been verified that the cross derivatives of the function Ω_3 are consistent, that is

$$
\frac{\partial^2 \Omega_3}{\partial q \partial p} = \frac{\partial^2 \Omega_3}{\partial p \partial q}, \qquad \frac{\partial^2 \Omega_3}{\partial q \partial \lambda} = \frac{\partial^2 \Omega_3}{\partial \lambda \partial q}, \qquad \frac{\partial^2 \Omega_3}{\partial p \partial \lambda} = \frac{\partial^2 \Omega_3}{\partial \lambda \partial p}.
$$

For this task, in addition to (A.3), the following relations have been used:

$$(U \pm V)_p = (U \mp V)_q, \qquad f_{qq}(f + g)_q = f_q(f + g)_{qq},$$

$$f_q(g + g_\lambda) = g_q(f + f_\lambda) \tag{A.4}$$

where

$$(U - V) = \tfrac{1}{2}(f_q g_\lambda - f_\lambda g_q) = (U + V)_{pq}.$$

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